Connection coefficients for Laguerre–Sobolev orthogonal polynomials

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Abstract

Laguerre–Sobolev polynomials are orthogonal with respect to an inner product of the form $\langle p,q\rangle_{\mathrm{S}} = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + \lambda\int_0^\infty p'(x)q'(x)\,d\mu(x)$, where $\alpha > -1$, $\lambda \geqslant 0$, and $p,q \in \mathbb{P}$, the linear space of polynomials with real coefficients. If $d\mu(x) = x^\alpha e^{-x}\,dx$, the corresponding sequence of monic orthogonal polynomials $\{Q_n^{(\alpha,\lambda)}(x)\}$ has been studied by Marcellán et al. (J. Comput. Appl. Math. 71 (1996) 245–265), while if $d\mu(x) = \delta(x)\,dx$ the sequence of monic orthogonal polynomials $\{L_n^{(\alpha)}(x;\lambda)\}$ was introduced by Koekoek and Meijer (SIAM J. Math. Anal. 24 (1993) 768–782). For each of these two families of Laguerre–Sobolev polynomials, here we give the explicit expression of the connection coefficients in their expansion as a series of standard Laguerre polynomials. The inverse connection problem of expanding Laguerre polynomials in series of Laguerre–Sobolev polynomials, and the connection problem relating two families of Laguerre–Sobolev polynomials with different parameters, are also considered.

Keywords Laguerre polynomials; Sobolev inner products; Connection coefficients

1. Introduction

Let $\{p_n(x)\}$ and $\{q_n(x)\}$ denote two (usually orthogonal) sequences of polynomials, with $\deg p_n(x) = \deg q_n(x) = n$. The connection problem consists in finding the coeffi-

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cients in the expansion of $q_n(x)$ in terms of the sequence $\{p_n(x)\}$,

$$q_n(x) = \sum_{k=0}^{n} c_{nk} p_k(x). \tag{1}$$

Each of the sequences $\{q_n(x)\}$ and $\{p_n(x)\}$ is a basis for $\mathbb{P}$, the linear space of polynomials with real coefficie ts. Likewise, the sets $\{q_0(x), q_1(x), \ldots, q_m(x)\}$ and $\{p_0(x), p_1(x), \ldots, p_m(x)\}$ are basis for $\mathbb{P}_m$, the subspace of polynomials of degree $\leqslant m$. Therefore, Eq. (1) corresponds to a change of basis in $\mathbb{P}_m$, and c_{nk} are the entries of the associated matrix. Several generalizations of this problem can be considered, such as replacing the variable x by a more complicated function in the argument of the polynomials.

Connection problems are often encountered both in pure and applied mathematics, and a wide variety of methods have been devised for computing the coeff cients c_{nk}, either in closed form (usually in terms of generalized hypergeometric series), or by means of recurrence relations (usually in the parameter k); see, e.g., reviews in [2, Section 7.1], [3, Lecture 7], [18], and references therein. For instance, let us consider the *monic* Laguerre polynomials $\{L_n^{(\alpha)}(x)\}$, which form an orthogonal sequence with respect to the inner product

$$\langle p, q\rangle = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\, dx, \quad \alpha > -1,$$

and have the explicit representation

$$L_n^{(\alpha)}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (k+\alpha+1)_{n-k}(-x)^k = (-1)^n(\alpha+1)_n\, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \,\middle|\, x\right).$$

It is well known (see, e.g., proofs in [2, Section 7.1] and [18]) that the connection formula relating two families of Laguerre polynomials of different parameters is

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}(\alpha-\beta)_{n-k} L_k^{(\beta)}(x). \tag{2}$$

As discussed in [18], inspection of (2) reveals that in this case the sign of the connection coefficient c_{nk} is $(-1)^{n-k}$ if $\alpha > \beta$ [9], while $c_{nk} \geqslant 0$ if $\alpha - \beta$ is a negative integer. On the other hand, if $\alpha - \beta < 0$ is not an integer, c_{nk} is nonnegative provided that $\beta - \alpha \geqslant n - k - 1$, so that all the connection coeff cients are nonnegative if $\beta - \alpha \geqslant n - 1$.

The above identity is just an example (perhaps the simplest one) of the many results on connection coeff cients that are known for the classical families of orthogonal polynomials. However, very little is known about connection problems involving nonclassical families, such as the generalizations of the classical families with Sobolev-type orthogonality. This kind of orthogonality appears in approximation theory when polynomial least square approximants are considered [14,17], as well as in harmonic analysis, where the Sobolev–Fourier series is more competitive than the standard one in order to analyze Gibbs phenomena [10,11], and in spectral methods for boundary value problems [6].

The aim of this paper is to solve the connection problems involving two different generalizations of Laguerre polynomials belonging to this class, which in recent years have deserved considerable attention in the literature. We recall that Laguerre–Sobolev polynomials are orthogonal with respect to an inner product of the form

$$\langle p, q\rangle_{\mathrm{S}} = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + \lambda \int_0^\infty p'(x)q'(x)\,d\mu(x), \tag{3}$$

where $\alpha > -1$, $\lambda \geqslant 0$, and $p, q \in \mathbb{P}$, the linear space of polynomials with real coefficients. If $d\mu(x) = x^\alpha e^{-x}\,dx$, the corresponding sequence of monic orthogonal polynomials $\{Q_n^{(\alpha,\lambda)}(x)\}$ has been studied in [15], while if $d\mu(x) = \delta(x)\,dx$ the sequence of monic orthogonal polynomials $\{L_n^{(\alpha)}(x;\lambda)\}$ was introduced in [13]. For each of these two families, here we give the explicit expression of the connection coefficients in their expansion as series of standard Laguerre polynomials, as well as the inverse expansions and the connection coefficients relating two families of Laguerre–Sobolev polynomials with different parameters. Problems involving the polynomials $\{Q_n^{(\alpha,\lambda)}(x)\}$ and $\{L_n^{(\alpha)}(x;\lambda)\}$ are dealt with in Sections 2 and 3, respectively. Finally, in Section 4 the results obtained are briefly summarized and several open problems are pointed out.

2. Connection problems involving the polynomials $Q_n^{(\alpha,\lambda)}(x)$

2.1. The polynomials $Q_n^{(\alpha,\lambda)}(x)$

Let $\mathcal{F}$ be the second-order linear differential operator defined by

$$\mathcal{F} = x\mathcal{I} - \lambda\mathcal{L}^{(\alpha-1)}, \tag{4}$$

where $\mathcal{I}$ is the identity operator, λ is a nonnegative real number, and $\mathcal{L}^{(\alpha-1)}$ denotes the Laguerre differential operator of parameter $\alpha - 1$. We recall that $\mathcal{L}^{(\alpha-1)}$ is defined as

$$\mathcal{L}^{(\alpha-1)} = (\alpha - x)\mathrm{D} + x\mathrm{D}^2,$$

where D and D^2 denote the differentiation operator and the second derivative operator, respectively. Let us also denote by $\{Q_n^{(\alpha,\lambda)}(x)\}$ a sequence of monic Laguerre–Sobolev polynomials, orthogonal with respect to the inner product

$$\langle p, q\rangle_{\mathrm{S}} = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + \lambda \int_0^\infty p'(x)q'(x)x^\alpha e^{-x}\,dx, \tag{5}$$

where $\alpha > -1$, $\lambda \geqslant 0$. In the literature, this kind of orthogonal polynomials are said to be continuous Laguerre–Sobolev orthogonal polynomials, taking into account that the measures involved in (5) are both nondiscrete.

In a recent work [15], Marcellán et al. proved that

$$\mathcal{F}Q_n^{(\alpha,\lambda)}(x) = L_{n+1}^{(\alpha-1)}(x) + \frac{q_n(\lambda)}{q_{n-1}(\lambda)}L_n^{(\alpha-1)}(x), \quad n \geqslant 1, \tag{6}$$

where the polynomials $q_n(\lambda)$ are defined by the three-term recurrence relation

$$q_n(\lambda) = \big[(\lambda+2)n + \alpha\big]q_{n-1}(\lambda) - n(n+\alpha-1)q_{n-2}(\lambda), \quad n \geqslant 2, \tag{7}$$

together with the initial conditions

$$q_0(\lambda) = 1, \qquad q_1(\lambda) = \lambda + \alpha + 1.$$

Equation (6) also holds when $n = 0$ if we set $q_{-1}(\lambda) = 1/\alpha$, since $\mathcal{F} Q_0^{(\alpha,\lambda)}(x) = x$. Thus we see that, if $\alpha \neq 0$, (7) is valid for $n \geqslant 1$ with the initial conditions $q_{-1}(\lambda) = 1/\alpha$, $q_0(\lambda) = 1$.

The polynomials $q_n(\lambda)$ can be expressed as [21]

$$q_n(\lambda) = (\alpha+1)_n \left[P_n^{(1-\alpha/2)}\left(\frac{\lambda}{2}+1; -\frac{\alpha}{2}, \frac{\alpha}{2}, \alpha\right) - \frac{1}{(\alpha+1)} P_{n-1}^{(1-\alpha/2)}\left(\frac{\lambda}{2}+1; -\frac{\alpha}{2}, \frac{\alpha}{2}, \alpha+1\right)\right], \tag{8}$$

where $P_n^{(\lambda)}(x; a, b, c)$ stands for the generalized Pollaczek polynomial (see, e.g., [7, p. 185]). In the particular case $\alpha = 0$, the Pollaczek polynomials reduce to Chebyshev polynomials of the second kind, and (8) simplifies to

$$q_n(\lambda) = \frac{n!}{2^{n+1}} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2+4\lambda}}\right)\left(\lambda + 2 + \sqrt{\lambda^2+4\lambda}\right)^n + \left(1 - \frac{\lambda}{\sqrt{\lambda^2+4\lambda}}\right)\left(\lambda + 2 - \sqrt{\lambda^2+4\lambda}\right)^n \right],$$

which can be written in a more compact form as

$$q_n(\lambda) = n! \, \frac{\cosh[(2n+1)\theta]}{\cosh\theta}, \quad \lambda \equiv 4\sinh^2\theta. \tag{9}$$

2.2. *Expansions of the Laguerre–Sobolev polynomials $Q_n^{(\alpha,\lambda)}(x)$ in series of Laguerre polynomials*

We begin by finding the explicit form of the coefficients $\mu_{n,j}$ in the expansion

$$Q_n^{(\alpha,\lambda)}(x) = \sum_{j=0}^{n} \mu_{n,j} L_j^{(\alpha-1)}(x). \tag{10}$$

To achieve this goal, we apply the operator $\mathcal{F}$ in (4) to both sides of Eq. (10),

$$\mathcal{F} Q_n^{(\alpha,\lambda)}(x) = \sum_{j=0}^{n} \mu_{n,j} \mathcal{F} L_j^{(\alpha-1)}(x). \tag{11}$$

Next we take into account that

$$\begin{aligned}\mathcal{F} L_j^{(\alpha-1)}(x) &= x L_j^{(\alpha-1)}(x) + \lambda j L_j^{(\alpha-1)}(x) \\ &= L_{j+1}^{(\alpha-1)}(x) + \big[(\lambda+2)j + \alpha\big] L_j^{(\alpha-1)}(x) + j(j+\alpha-1) L_{j-1}^{(\alpha-1)}(x),\end{aligned}$$

where the second expression in the right-hand side is obtained from the first one by means of the well-known three-term recurrence relation satisfied by monic Laguerre polynomials,

$$L_{j+1}^{(\alpha-1)}(x) + (2j+\alpha-x) L_j^{(\alpha-1)}(x) + j(j+\alpha-1) L_{j-1}^{(\alpha-1)}(x) = 0.$$

Equation (11) can thus be written as

$$
\begin{aligned}
&\mathcal{F} Q_n^{(\alpha,\lambda)}(x) \\
&\quad = \mu_{n,n} L_{n+1}^{(\alpha-1)}(x) + \bigl\{\mu_{n,n-1} + \bigl[(\lambda+2)n+\alpha\bigr]\mu_{n,n}\bigr\} L_n^{(\alpha-1)}(x) \\
&\qquad + \sum_{j=0}^{n-1} \bigl\{\mu_{n,j-1} + \bigl[(\lambda+2)j+\alpha\bigr]\mu_{n,j} + (j+1)(j+\alpha)\mu_{n,j+1}\bigr\} L_j^{(\alpha-1)}(x),
\end{aligned}
$$

where it is understood that $\mu_{n,-1} \equiv 0$. Comparison of this result with Eq. (6) shows that

$$
\mu_{n,n} = 1, \qquad \mu_{n,n-1} = \frac{q_n(\lambda)}{q_{n-1}(\lambda)} - \bigl[(\lambda+2)n+\alpha\bigr], \tag{12}
$$

while for $0 \leqslant j \leqslant n-1$ it holds the three-term recurrence relation

$$
\mu_{n,j-1} + \bigl[(\lambda+2)j+\alpha\bigr]\mu_{n,j} + (j+1)(j+\alpha)\mu_{n,j+1} = 0. \tag{13}
$$

Use of Eq. (7) enables us to write the second equation of (12) in the alternative form

$$
\mu_{n,n-1} = -n(n+\alpha-1)\frac{q_{n-2}(\lambda)}{q_{n-1}(\lambda)},
$$

which turns out to be more convenient in order to use (13) as a backward recurrence relation. Computing $\mu_{n,n-2}, \mu_{n,n-3}, \ldots$ by means of this relation, and using (7) in each step to simplify the resulting expression, one is naturally led to state the Ansatz,

$$
\mu_{n,j} = (-1)^{n-j} \frac{n!\,\Gamma(n+\alpha)}{j!\,\Gamma(j+\alpha)} \frac{q_{j-1}(\lambda)}{q_{n-1}(\lambda)}, \tag{14}
$$

whose validity can be readily checked by direct substitution into (13). The indetermination arising in the right-hand side of this equation when $\alpha = 0$, $j = 0$, $n \neq 0$ can be solved recalling that $q_{-1}(\lambda) = 1/\alpha$, hence

$$
\mu_{n,0} = \frac{(-1)^n n!\,\Gamma(n+\alpha)}{\Gamma(\alpha+1) q_{n-1}(\lambda)} = -\mu_{n,1}, \quad n \geqslant 1, \tag{15}
$$

which remains meaningful as $\alpha \to 0$. When $\alpha = 0$, use of Eq. (9) yields the simple formula

$$
\mu_{n,j} = (-1)^{n-j} \frac{n! \cosh[(2j-1)\theta]}{j! \cosh[(2n-1)\theta]}, \tag{16}
$$

and taking into account that $L_j^{(-1)}(0) = 0$ for $j \geqslant 1$, substitution of (16) into (10) with $x = 0$ gives the remarkable special value

$$
Q_n^{(0,\lambda)}(0) = \frac{(-1)^n n! \cosh\theta}{\cosh[(2n-1)\theta]}.
$$

Combining the connection formula for the standard Laguerre polynomials Eq. (2) with (10), we f nd the expansion of Laguerre–Sobolev polynomials in series of Laguerre polynomials with arbitrary parameters,

$$
Q_n^{(\alpha,\lambda)}(x) = \sum_{j=0}^{n} \left[\sum_{k=0}^{n-j} \mu_{n,j+k} \binom{j+k}{j} (-1)^k (\alpha-\beta)_k \right] L_j^{(\beta-1)}(x). \tag{17}
$$

In particular, for $\beta = \alpha + 1$, the connection formula (2) reads

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x), \tag{18}$$

and, recalling (15), Eq. (17) reduces to

$$Q_n^{(\alpha,\lambda)}(x) = L_n^{(\alpha)}(x) + \sum_{j=1}^{n-1}\big[\mu_{n,j} + (j+1)\mu_{n,j+1}\big]L_j^{(\alpha)}(x). \tag{19}$$

This expression is useful to f nd explicit formulas for the Laguerre–Sobolev polynomials of lowest degree,

$$\begin{aligned}
Q_0^{(\alpha,\lambda)}(x) &= L_0^{(\alpha)}(x) = 1,\\
Q_1^{(\alpha,\lambda)}(x) &= L_1^{(\alpha)}(x) = x - \alpha - 1,\\
Q_2^{(\alpha,\lambda)}(x) &= L_2^{(\alpha)}(x) + \frac{2\lambda}{(\lambda+\alpha+1)}L_1^{(\alpha)}(x)\\
&= x^2 - 2(\alpha+1)\left(\frac{\lambda+\alpha+2}{\lambda+\alpha+1}\right)x + (\alpha+1)\left(\frac{\alpha\lambda+(\alpha+1)(\alpha+2)}{\lambda+\alpha+1}\right).
\end{aligned}$$

In the particular case $\alpha = 0$, using Eq. (16), (19) takes the form

$$Q_n^{(0,\lambda)}(x) = L_n^{(0)}(x) + \frac{2n!\sinh\theta}{\cosh[(2n-1)\theta]}\sum_{j=1}^{n-1}\frac{(-1)^{n-1-j}\sinh(2j\theta)}{j!}L_j^{(0)}(x).$$

2.3. Expansions of Laguerre polynomials in series of the Laguerre–Sobolev polynomials $Q_n^{(\alpha,\lambda)}(x)$

From Eqs. (14) and (15) we see that the connection coeff cients $\mu_{n,j}$ can be factorized as

$$\mu_{n,j} = \frac{\mu_{n,0}}{\mu_{j,0}},$$

so that the connection formula (10) reads

$$Q_n^{(\alpha,\lambda)}(x) = \mu_{n,0}\sum_{j=0}^{n}\frac{1}{\mu_{j,0}}L_j^{(\alpha-1)}(x).$$

Therefrom it readily follows the inverse connection formula [15]

$$\begin{aligned}
L_n^{(\alpha-1)}(x) &= Q_n^{(\alpha,\lambda)}(x) - \frac{\mu_{n,0}}{\mu_{n-1,0}}Q_{n-1}^{(\alpha,\lambda)}(x) = Q_n^{(\alpha,\lambda)}(x) - \mu_{n,n-1}Q_{n-1}^{(\alpha,\lambda)}(x)\\
&= Q_n^{(\alpha,\lambda)}(x) + n(n+\alpha-1)\frac{q_{n-2}(\lambda)}{q_{n-1}(\lambda)}Q_{n-1}^{(\alpha,\lambda)}(x),
\end{aligned} \tag{20}$$

which using (18) can be written in the equivalent form

$$L_n^{(\alpha)}(x) + nL_{n-1}^{(\alpha)}(x) = Q_n^{(\alpha,\lambda)}(x) - \mu_{n,n-1}Q_{n-1}^{(\alpha,\lambda)}(x). \tag{21}$$

Combining Eq. (20) with (2), we fi d its generalization to arbitrary parameters of the involved polynomials,

$$L_n^{(\beta-1)}(x) = \sum_{j=0}^{n} (-1)^{n-j} \left[\binom{n}{j} (\beta-\alpha)_{n-j} + \mu_{j+1,j} \binom{n}{j+1} (\beta-\alpha)_{n-j-1} \right] Q_j^{(\alpha,\lambda)}(x). \tag{22}$$

In particular, for $\beta = \alpha + 1$, we have

$$L_n^{(\alpha)}(x) = n! \sum_{j=1}^{n} \frac{(-1)^{n-j}}{j!} \left(1 + \frac{\mu_{j+1,j}}{j+1} \right) Q_j^{(\alpha,\lambda)}(x),$$

which for $\alpha = 0$, using Eq. (16), takes the form

$$L_n^{(0)}(x) = 2n! \sinh\theta \sum_{j=1}^{n} \frac{(-1)^{n-j} \sinh(2j\theta)}{j! \cosh[(2j+1)\theta]} Q_j^{(0,\lambda)}(x).$$

It is worth noting that by means of Eqs. (20) and (6), which using (14) can be written as

$$\mathcal{F} Q_n^{(\alpha,\lambda)}(x) = L_{n+1}^{(\alpha-1)}(x) - \frac{(n+1)(n+\alpha)}{\mu_{n+1,n}} L_n^{(\alpha-1)}(x),$$

we can express the action of $\mathcal{F}$ on the Laguerre–Sobolev polynomial $Q_n^{(\alpha,\lambda)}(x)$ in terms of polynomials of the same kind [15],

$$\begin{aligned} \mathcal{F} Q_n^{(\alpha,\lambda)}(x) &= Q_{n+1}^{(\alpha,\lambda)}(x) - \frac{\{[\mu_{n+1,n}]^2 + (n+1)(n+\alpha)\}}{\mu_{n+1,n}} Q_n^{(\alpha,\lambda)}(x) \\ &\quad + (n+1)(n+\alpha) \frac{\mu_{n,n-1}}{\mu_{n+1,n}} Q_{n-1}^{(\alpha,\lambda)}(x) \\ &= Q_{n+1}^{(\alpha,\lambda)}(x) + \frac{\{[q_n(\lambda)]^2 + (n+1)(n+\alpha)[q_{n-1}(\lambda)]^2\}}{q_{n-1}(\lambda) q_n(\lambda)} Q_n^{(\alpha,\lambda)}(x) \\ &\quad + n(n+\alpha-1) \frac{q_n(\lambda) q_{n-2}(\lambda)}{[q_{n-1}(\lambda)]^2} Q_{n-1}^{(\alpha,\lambda)}(x). \end{aligned}$$

2.4. Connection formula relating two families of Laguerre–Sobolev polynomials $Q_n^{(\alpha,\lambda)}(x)$ with different parameters

Now let us consider the sequence of monic polynomials $\{Q_n^{(\beta,\mu)}(x)\}$, orthogonal with respect to the inner product

$$\langle p, q \rangle_{\mathrm{S}} = \int_0^\infty p(x) q(x) x^\beta e^{-x} \, dx + \mu \int_0^\infty p'(x) q'(x) x^\beta e^{-x} \, dx.$$

For these polynomials we write Eqs. (10) and (20) as

$$Q_n^{(\beta,\mu)}(x) = \sum_{j=0}^{n} \nu_{n,j} L_j^{(\beta-1)}(x),$$
$$L_n^{(\beta-1)}(x) = Q_n^{(\beta,\mu)}(x) - \nu_{n,n-1} Q_{n-1}^{(\beta,\mu)}(x). \tag{23}$$

The second of these equations, together with (17), enables us to f nd a connection formula relating two arbitrary sets of Laguerre–Sobolev polynomials,

$$\begin{aligned} &Q_n^{(\alpha,\lambda)}(x) \\ &\quad = \sum_{j=0}^{n} \left\{ \sum_{k=0}^{n-j} (-1)^k \mu_{n,j+k} \left[\binom{j+k}{j} (\alpha-\beta)_k + \nu_{j+1,j} \binom{j+k}{j+1} (\alpha-\beta)_{k-1} \right] \right\} \\ &\qquad \times Q_j^{(\beta,\mu)}(x). \end{aligned} \tag{24}$$

Alternatively, we can use the f rst equation in (23) together with (22), which leads to the same result. In the particular case $\alpha = \beta$, the previous identity simplifie to

$$Q_n^{(\alpha,\lambda)}(x) = Q_n^{(\alpha,\mu)}(x) + \sum_{j=0}^{n-1} (\mu_{n,j} - \mu_{n,j+1}\nu_{j+1,j}) Q_j^{(\alpha,\mu)}(x),$$

and, for $\alpha = 0$,

$$\begin{aligned} &Q_n^{(0,\lambda)}(x) \\ &\quad = Q_n^{(0,\mu)}(x) + \frac{n!}{\cosh[(2n-1)\theta]} \sum_{j=1}^{n-1} \frac{(-1)^{n-j}}{j!} \\ &\qquad \times \frac{\{\cosh[(2j-1)\theta]\cosh[(2j+1)\phi] - \cosh[(2j+1)\theta]\cosh[(2j-1)\phi]\}}{\cosh[(2j+1)\phi]} \\ &\qquad \times Q_j^{(0,\mu)}(x), \end{aligned}$$

where we have used Eq. (16) and its counterpart for the polynomials $Q_n^{(0,\mu)}(x)$,

$$\nu_{n,j} = (-1)^{n-j} \frac{n! \cosh[(2j-1)\phi]}{j! \cosh[(2n-1)\phi]}, \qquad \mu \equiv 4\sinh^2\phi.$$

3. Connection problems involving the polynomials $L_n^{(\alpha)}(x;\lambda)$

3.1. The polynomials $L_n^{(\alpha)}(x;\lambda)$

Denote $\{L_n^{(\alpha)}(x;\lambda)\}$ the sequence of monic polynomials orthogonal with respect to the inner product

$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}\,dx + \lambda p'(0)q'(0). \tag{25}$$

They are said to be discrete Laguerre–Sobolev orthogonal polynomials, taking into account that the measure involving derivatives is a discrete one. From results in Ref. [1], the following expression can be deduced for these polynomials in terms of the standard Laguerre polynomials:

$$L_n^{(\alpha)}(x;\lambda) = L_n^{(\alpha)}(x) - \frac{1}{\alpha_n} K_{n-1}^{(0,1)}(x,0), \quad \alpha_n \equiv \frac{1+\lambda K_{n-1}^{(1,1)}(0,0)}{\lambda [L_n^{(\alpha)}(0)]'}, \tag{26}$$

where $K_n^{(r,s)}(x,y)$ denotes the generalized kernels

$$K_n^{(r,s)}(x,y) = \sum_{j=0}^{n} \frac{[L_j^{(\alpha)}(x)]^{(r)}[L_j^{(\alpha)}(y)]^{(s)}}{\|L_j^{(\alpha)}\|_\alpha^2}. \tag{27}$$

Recalling the explicit expressions

$$\|L_j^{(\alpha)}\|_\alpha^2 = j!\,\Gamma(j+\alpha+1) \tag{28}$$

and

$$\big[L_j^{(\alpha)}(0)\big]' = \frac{(-1)^{j-1} j\,\Gamma(j+\alpha+1)}{\Gamma(\alpha+2)}, \tag{29}$$

we readily fin that

$$K_{n-1}^{(0,1)}(x,0) = \frac{1}{\Gamma(\alpha+2)} \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{(j-1)!} L_j^{(\alpha)}(x). \tag{30}$$

Combining this result with the connection formula for Laguerre polynomials (2) and simplifying the resulting expression by means of the summation formula

$$\sum_{k=0}^{m} \frac{(a)_k}{k!} = \frac{(a+1)_m}{m!}, \tag{31}$$

which can be easily proved by induction on m, we f nd that

$$\begin{aligned} K_{n-1}^{(0,1)}(x,0) = {} & \frac{1}{(n-1)!\,\Gamma(\alpha+2)} \\ & \times \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{k-1} (\alpha-\beta+2)_{n-k-2} \\ & \times \big[(\alpha-\beta)(n-1)+k\big] L_k^{(\beta)}(x). \end{aligned} \tag{32}$$

In the particular case $\beta=\alpha+2$, only the last two terms of the sum in the right-hand side are nonzero, and we have

$$K_{n-1}^{(0,1)}(x,0) = \frac{(-1)^n}{(n-2)!\,\Gamma(\alpha+2)} \big[L_{n-1}^{(\alpha+2)}(x) + n L_{n-2}^{(\alpha+2)}(x)\big]. \tag{33}$$

Differentiating this equation with respect to x and setting $x=0$, a straightforward calculation using (29) leads to

$$K_{n-1}^{(1,1)}(0,0) = \frac{\Gamma(n+\alpha+1)[(\alpha+2)n-(\alpha+1)]}{(n-2)!\,\Gamma(\alpha+2)\Gamma(\alpha+4)}. \tag{34}$$

Two alternative derivations of this result, based respectively on the Christoffel–Darboux formula for orthogonal polynomials and the representation as hypergeometric series of the generalized kernels, are given in Appendix A. Collecting Eqs. (29) and (34), we can write the explicit expression of the constants α_n in (26),

$$\alpha_n = \frac{(-1)^{n-1}\Gamma(\alpha+2)}{\lambda n\Gamma(n+\alpha+1)}\left[1+\lambda\frac{\Gamma(n+\alpha+1)[(\alpha+2)n-(\alpha+1)]}{(n-2)!\,\Gamma(\alpha+2)\Gamma(\alpha+4)}\right]. \tag{35}$$

3.2. Expansions of the Laguerre–Sobolev polynomials $L_n^{(\alpha)}(x;\lambda)$ in series of Laguerre polynomials

Equation (26) is already a connection formula, giving the expansion of the polynomials $\{L_n^{(\alpha)}(x;\lambda)\}$ in series of Laguerre polynomials with the same parameter $\{L_k^{(\alpha)}(x)\}$. With account of (30), it can be written more explicitly as

$$L_n^{(\alpha)}(x;\lambda) = L_n^{(\alpha)}(x) - \frac{1}{\alpha_n}\frac{1}{\Gamma(\alpha+2)}\sum_{j=1}^{n-1}\frac{(-1)^{j-1}}{(j-1)!}L_j^{(\alpha)}(x). \tag{36}$$

In particular, for the Laguerre–Sobolev polynomials of lowest degree,

$$\begin{aligned}
L_0^{(\alpha)}(x;\lambda) &= L_0^{(\alpha)}(x) = 1,\\
L_1^{(\alpha)}(x;\lambda) &= L_1^{(\alpha)}(x) = x-\alpha-1,\\
L_2^{(\alpha)}(x;\lambda) &= L_2^{(\alpha)}(x) - \frac{1}{\alpha_2}\frac{1}{\Gamma(\alpha+2)}L_1^{(\alpha)}(x) = L_2^{(\alpha)}(x) + \frac{2\lambda(\alpha+2)}{[\Gamma(\alpha+2)+\lambda]}L_1^{(\alpha)}(x)\\
&= x^2 - \frac{2\Gamma(\alpha+3)}{[\Gamma(\alpha+2)+\lambda]}x + (\alpha+1)(\alpha+2)\left(\frac{\Gamma(\alpha+2)-\lambda}{\Gamma(\alpha+2)+\lambda}\right).
\end{aligned}$$

Substitution of Eqs. (2) and (32) into (26) enables us to f nd the general connection formula corresponding to the expansion of $\{L_n^{(\alpha)}(x;\lambda)\}$ in series of Laguerre polynomials with arbitrary parameter $\{L_k^{(\beta)}(x)\}$,

$$\begin{aligned}
L_n^{(\alpha)}(x;\lambda) = L_n^{(\beta)}(x) &+ \sum_{k=0}^{n-1}\binom{n}{k}(-1)^{n-k}(\alpha-\beta)_{n-k}\\
&\times\left(1+\frac{(-1)^n(n-k)[(\alpha-\beta)(n-1)+k]}{\alpha_n(\alpha-\beta+1)(\alpha-\beta)n!\,\Gamma(\alpha+2)}\right)L_k^{(\beta)}(x).
\end{aligned} \tag{37}$$

In the particular case $\beta=\alpha+2$, Eqs. (2) and (32) reduce, respectively, to

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+2)}(x) + 2nL_{n-1}^{(\alpha+2)}(x) + n(n-1)L_{n-2}^{(\alpha+2)}(x)$$

and (33). Thus Eq. (37) simplif es to

$$\begin{aligned}
L_n^{(\alpha)}(x;\lambda) = L_n^{(\alpha+2)}(x) &+ \left(2n - \frac{(-1)^n}{\alpha_n(n-2)!\,\Gamma(\alpha+2)}\right)L_{n-1}^{(\alpha+2)}(x)\\
&+ n\left((n-1) - \frac{(-1)^n}{\alpha_n(n-2)!\,\Gamma(\alpha+2)}\right)L_{n-2}^{(\alpha+2)}(x).
\end{aligned} \tag{38}$$

It is convenient to introduce the constants

$$\lambda_n \equiv \frac{1+\lambda K_n^{(1,1)}(0,0)}{1+\lambda K_{n-1}^{(1,1)}(0,0)} = 1 - \frac{(-1)^n}{\alpha_n (n-1)!\,\Gamma(\alpha+2)}, \tag{39}$$

where the second expression in the right-hand side is obtained from the f rst one by using (27)–(29) together with the second equation in (26). Equation (38) can thus be written as

$$L_n^{(\alpha)}(x;\lambda) = L_n^{(\alpha+2)}(x) + \big[(n+1)+(n-1)\lambda_n\big]L_{n-1}^{(\alpha+2)}(x) + n(n-1)\lambda_n L_{n-2}^{(\alpha+2)}(x). \tag{40}$$

Note from the f rst expression in (39) that $\lambda_n > 1$, since $K_n^{(1,1)}(0,0) > K_{n-1}^{(1,1)}(0,0)$. Thus we see, in particular, that the connection coeff cients in (40) are all positive. From Eqs. (38) and (40), recalling that

$$L_n^{(\alpha)}(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}, \tag{41}$$

we can easily fin the special value

$$\begin{aligned} L_n^{(\alpha)}(0;\lambda) &= (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\left(1+\frac{(-1)^n}{\alpha_n (n-2)!\,\Gamma(\alpha+3)}\right) \\ &= (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+3)}(\alpha+1)\big[(n+\alpha+1)-(n-1)\lambda_n\big]. \end{aligned}$$

The fact that, in the expansion

$$L_n^{(\alpha)}(x;\lambda) = \sum_{j=0}^{n} \rho_{n,j} L_j^{(\alpha+2)}(x),$$

$\rho_{n,j} = 0$ for $0 \leqslant j \leqslant n-3$ can be easily proved by taking into account that

$$\rho_{n,j} = \frac{\int_0^\infty L_n^{(\alpha)}(x;\lambda) L_j^{(\alpha+2)}(x) x^{\alpha+2} e^{-x}\,dx}{\|L_j^{(\alpha+2)}\|_{\alpha+2}^2} = \frac{\langle L_n^{(\alpha)}(x;\lambda), x^2 L_j^{(\alpha+2)}(x)\rangle_{\mathrm{S}}}{\|L_j^{(\alpha+2)}\|_{\alpha+2}^2}.$$

A similar method can be applied to derive in an alternative way the explicit expressions of $\rho_{n,n-1}$ and $\rho_{n,n-2}$ (see Appendix A).

3.3. Expansions of Laguerre polynomials in series of the Laguerre–Sobolev polynomials $L_n^{(\alpha)}(x;\lambda)$

Equation (36) can be written in the alternative form

$$\alpha_n\big(L_n^{(\alpha)}(x) - L_n^{(\alpha)}(x;\lambda)\big) = \frac{1}{\Gamma(\alpha+2)} \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{(j-1)!} L_j^{(\alpha)}(x),$$

from which one can readily fin that

$$
\begin{aligned}
&\alpha_{n+1}\big(L_{n+1}^{(\alpha)}(x) - L_{n+1}^{(\alpha)}(x;\lambda)\big) - \alpha_n\big(L_n^{(\alpha)}(x) - L_n^{(\alpha)}(x;\lambda)\big)\\
&\quad = \frac{(-1)^{n-1}}{(n-1)!\,\Gamma(\alpha+2)} L_n^{(\alpha)}(x) = \alpha_n(\lambda_n - 1) L_n^{(\alpha)}(x).
\end{aligned}
$$

Thus we see that it holds the identity

$$
L_{n+1}^{(\alpha)}(x;\lambda) - \frac{\alpha_n}{\alpha_{n+1}} L_n^{(\alpha)}(x;\lambda) = L_{n+1}^{(\alpha)}(x) - \frac{\alpha_n}{\alpha_{n+1}} \lambda_n L_n^{(\alpha)}(x). \tag{42}
$$

Notice that there is a similarity between this equation and (21), despite the fact that the inner products involved are quite different. Combination of the first expression for λ_n in (39) with the second equation in (26) yields

$$
\frac{\alpha_n}{\alpha_{n+1}} = \frac{1}{\lambda_n}\frac{[L_{n+1}^{(\alpha)}(0)]'}{[L_n^{(\alpha)}(0)]'} = -\frac{(n+1)(n+\alpha+1)}{n\lambda_n},
$$

where the second expression in the right-hand side is obtained from the first one by using (29). This result enables us to write Eq. (42) in the more explicit form

$$
\begin{aligned}
&L_{n+1}^{(\alpha)}(x;\lambda) + \frac{(n+1)(n+\alpha+1)}{n\lambda_n} L_n^{(\alpha)}(x;\lambda)\\
&\quad = L_{n+1}^{(\alpha)}(x) + \frac{(n+1)(n+\alpha+1)}{n} L_n^{(\alpha)}(x).
\end{aligned}
$$

Let us now introduce the new constants

$$
b_n \equiv \prod_{k=1}^{n}\left(-\frac{(k+1)(k+\alpha+1)}{k}\right) = \frac{(-1)^n(n+1)\Gamma(n+\alpha+2)}{\Gamma(\alpha+2)}.
$$

In terms of the polynomials

$$
R_n(x) \equiv \frac{1}{b_{n-1}} L_n^{(\alpha)}(x), \quad n \geqslant 1, \tag{43}
$$

Eq. (42) reads

$$
R_{k+1}(x) - R_k(x) = \frac{1}{b_k}\left[L_{k+1}^{(\alpha)}(x;\lambda) - \frac{\alpha_k}{\alpha_{k+1}} L_k^{(\alpha)}(x;\lambda)\right].
$$

Summing this equation for k running from 1 to $n-1$, we get,

$$
\begin{aligned}
R_n(x) - R_1(x) = {}& \frac{1}{b_{n-1}} L_n^{(\alpha)}(x;\lambda) - \frac{1}{b_1}\frac{\alpha_1}{\alpha_2} L_1^{(\alpha)}(x;\lambda)\\
&+ \sum_{j=2}^{n-1}\left(\frac{1}{b_{j-1}} - \frac{1}{b_j}\frac{\alpha_j}{\alpha_{j+1}}\right) L_j^{(\alpha)}(x;\lambda).
\end{aligned}
$$

Recalling Eq. (43) and taking into account that, in particular, $R_1(x) = L_1^{(\alpha)}(x) = L_1^{(\alpha)}(x;\lambda)$, the previous equation simplifies to

$$
L_n^{(\alpha)}(x) = L_n^{(\alpha)}(x;\lambda) + b_{n-1}\sum_{j=1}^{n-1} \gamma_j L_j^{(\alpha)}(x;\lambda), \tag{44}
$$

where

$$\gamma_n \equiv \frac{1}{b_{n-1}} - \frac{1}{b_n}\frac{\alpha_n}{\alpha_{n+1}} = \frac{1}{b_{n-1}}\frac{(\lambda_n - 1)}{\lambda_n} = \frac{(\lambda_n - 1)}{b_n}\frac{\alpha_n}{\alpha_{n+1}}$$
$$= -\frac{1}{\alpha_{n+1}(n+1)(n-1)!\,\Gamma(n+\alpha+2)}.$$

We have thus obtained the inverse connection formula giving the expansion of Laguerre polynomials $L_n^{(\alpha)}(x)$ in series of the $L_n^{(\alpha)}(x;\lambda)$. The generalization of this result to polynomials with arbitrary parameters can again be obtained from its combination with (2), which leads to

$$L_n^{(\beta)}(x)$$
$$= \sum_{k=0}^{n}\left\{\binom{n}{k}(-1)^{n-k}(\beta-\alpha)_{n-k} + \gamma_k \sum_{j=k+1}^{n}\binom{n}{j}(-1)^{n-j}(\beta-\alpha)_{n-j}b_{j-1}\right\}$$
$$\times L_k^{(\alpha)}(x;\lambda) \tag{45}$$

(here it is understood that $\gamma_0 \equiv 0$, and we follow the usual convention of taking the value of empty sums to be zero).

3.4. Connection formula relating two families of Laguerre–Sobolev polynomials $L_n^{(\alpha)}(x;\lambda)$ with different parameters

Now let us consider the sequence of monic polynomials $\{L_n^{(\beta)}(x;\mu)\}$, orthogonal with respect to the inner product

$$\langle p, q\rangle_{\mathrm{S}} = \int_0^\infty p(x)q(x)x^\beta e^{-x}\,dx + \mu p'(0)q'(0).$$

The counterpart of Eq. (37) for these polynomials can be written as

$$L_n^{(\beta)}(x;\mu) = L_n^{(\alpha)}(x) + \sum_{k=0}^{n-1} d_{n,k} L_k^{(\alpha)}(x),$$
$$d_{n,k} \equiv \binom{n}{k}(-1)^{n-k}(\beta-\alpha)_{n-k}\left(1 + \frac{(-1)^n(n-k)[(\beta-\alpha)(n-1)+k]}{\beta_n(\beta-\alpha+1)(\beta-\alpha)n!\,\Gamma(\beta+2)}\right),$$

where the constants β_n are define as the α_n in (26) or (35) replacing α and λ by β and μ, respectively. Substitution of (44) into the previous equation enables us to fin the connection formula relating two sets of Laguerre–Sobolev polynomials with different parameters,

$$L_n^{(\beta)}(x;\mu) = L_n^{(\alpha)}(x;\lambda) + \sum_{k=0}^{n-1}\left[d_{n,k} + \gamma_k\left(b_{n-1} + \sum_{j=k+1}^{n-1} d_{n,j}b_{j-1}\right)\right]L_k^{(\alpha)}(x;\lambda). \tag{46}$$

Substantial simplificati ns arise in the particular cases $\alpha = \beta$, where according to (36),

$$d_{n,k} = \frac{(-1)^k}{\beta_n (k-1)!\, \Gamma(\alpha+2)},$$

and $\alpha = \beta + 2$, where according to (38) or (40), $d_{n,k} = 0$ unless $k = n-1, n-2$.

4. Summary and open problems

In this paper, two families of polynomials orthogonal with respect to a Sobolev inner product of the form (3), namely the polynomials $\{Q_n^{(\alpha,\lambda)}(x)\}$ and $L_n^{(\alpha)}(x;\lambda)$, have been considered. We have emphasized the analysis of their corresponding connection coeff -cients, whose explicit expressions have been derived using known properties of Laguerre–Sobolev polynomials together with the classical connection formula for standard Laguerre polynomials (2). The main results given can be summarized in the following diagrams:

$$\begin{tikzcd} L_n^{(\alpha)}(x) \arrow[r, "(2)"] & L_n^{(\beta)}(x) \arrow[d, "(22)"] \\ Q_n^{(\beta,\mu)}(x) \arrow[u, "(17)"] & Q_n^{(\alpha,\lambda)}(x) \arrow[l, "(24)"'] \end{tikzcd} \qquad \begin{tikzcd} L_n^{(\alpha)}(x) \arrow[r, "(2)"] & L_n^{(\beta)}(x) \arrow[d, "(45)"] \\ L_n^{(\beta)}(x;\mu) \arrow[u, "(37)"] & L_n^{(\alpha)}(x;\lambda) \arrow[l, "(46)"'] \end{tikzcd}$$

where $p_n(x) \to q_n(x)$ stands for the expansion of $p_n(x)$ in series of the $\{q_n(x)\}$.

Connection problems involving the polynomials $\{Q_n^{(\alpha,\lambda)}(x)\}$ have been considered here for the f rst time. For the polynomials $L_n^{(\alpha)}(x;\lambda)$, this is the f rst time when connection problems are considered without making use of the hypergeometric representation given for these polynomials in [12,13], which was the basis of the approach previously followed in [19]. For further work on this subject, the following four problems are suggested.

(1) For the harmonic analysis with respect to orthogonal polynomial sequences it is important to know the sign properties of the connection coeff cients, especially whether they are positive or nonnegative (cf. the survey papers by Askey [3] and Gasper [9]). For instance, as discussed in [2, Section 7.4], the nonnegativity of some connection coeff cients for Gegenbauer polynomials is useful in the proof of the positivity of a certain hypergeometric function, a result which played a signif cant role in the f rst proof of the celebrated Bieberbach conjecture from complex analysis, due to de Branges [5]. It has also been recently shown [8] that sign properties of connection coeff cients are closely related to the behavior of the zeros of the involved polynomials. As pointed out in Section 1, for the standard Laguerre case sign properties of the connection coeff cients can be readily found by inspection of (2). Thus we can pose the problem of analyzing whether these sign properties are preserved for the Laguerre–Sobolev polynomials.

(2) From Eqs. (21) and (42), it arises as a natural question to characterize the measures μ in (3) such that the corresponding orthogonal sequences $\{q_n(x)\}$ of monic polynomials are related to Laguerre polynomials by an identity of the form

$$q_n(x) + s_n q_{n-1}(x) = L_n^{(\alpha)}(x) + t_n L_{n-1}^{(\alpha)}(x).$$

Notice that we know at least two solutions for this kind of measures, and our interest is to describe the complete set of them. The main application of this result is related to the analysis of the relative asymptotics of such polynomials in terms of Laguerre polynomials.

(3) If we consider a Gegenbauer–Sobolev inner product,

$$\langle p, q\rangle_{\mathrm{S}} = \int_{-1}^{1} p(x)q(x)(1-x^2)^{\alpha-1/2}\,dx + \lambda \int_{-1}^{1} p'(x)q'(x)\,d\mu(x), \tag{47}$$

where $\alpha > -1/2$, $\lambda \geqslant 0$, it is well known [16] that if $d\mu(x) = (1-x^2)^{\alpha-1/2}\,dx$, then there exists a second-order differential operator $\mathcal{G}$ which is symmetric with respect to (47). Notice that in the Laguerre–Sobolev case the operator $\mathcal{F}$ defi ed by (4) is symmetric with respect to (5). Furthermore, in Ref. [11] it is proved that such Gegenbauer–Sobolev orthogonal polynomials satisfy an algebraic relation very similar to (21). Thus we conclude that in such a case we can proceed as in our present contribution, and this work is in progress at present time.

(4) Extensions of the inner product (3) involving higher-order derivatives have been considered in [12,16]. The analysis of the connection coeff cients for the corresponding sequences of orthogonal polynomials remains an open problem.

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Appendix A. Alternative derivations of some results

A.1. Calculation of $K_{n-1}^{(1,1)}(0,0)$ from the Christoffel–Darboux formula

In order to deduce the expression of $K_{n-1}^{(1,1)}(0,0)$ for general orthogonal polynomials, we shall use the Christoffel–Darboux formula,

$$\|P_{n-1}\|^2 K_{n-1}(x,y) = \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{x-y},$$

where $K_n(x,y) \equiv K_n^{(0,0)}(x,y)$. Differentiating with respect to y, we f nd that

$$\begin{aligned}\|P_{n-1}\|^2 K_{n-1}^{(0,1)}(x,y) = {} & \frac{P_n(x)P'_{n-1}(y) - P'_n(y)P_{n-1}(x)}{x-y} \\ & + \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{(x-y)^2}.\end{aligned}$$

In particular, setting $y = 0$,

$$\|P_{n-1}\|^2 K_{n-1}^{(0,1)}(x,0) = \frac{P_n(x)P'_{n-1}(0) - P'_n(0)P_{n-1}(x)}{x} + \frac{P_n(x)P_{n-1}(0) - P_n(0)P_{n-1}(x)}{x^2}.$$

Thus we see that in the Taylor expansions of $P_n(x)$ and $P_{n-1}(x)$ only the terms of degree x^2 and x^3 give a nonvanishing contribution to $\|P_{n-1}\|^2 K_{n-1}^{(1,1)}(0,0)$, and we f nally get

$$\|P_{n-1}\|^2 K_{n-1}^{(1,1)}(0,0) = \frac{1}{2}\big(P''_n(0)P'_{n-1}(0) - P'_n(0)P''_{n-1}(0)\big) + \frac{1}{6}\big(P'''_n(0)P_{n-1}(0) - P_n(0)P'''_{n-1}(0)\big).$$

For Laguerre polynomials, using identities (28) and (41), together with

$$\big[L_n^{(\alpha)}(x)\big]^{(r)} = \frac{n!}{(n-r)!} L_{n-r}^{(\alpha+r)}(x), \tag{A.1}$$

a somewhat lengthy but straightforward calculation leads to (34).

A.2. Calculation of $K_{n-1}^{(1,1)}(0,0)$ from the hypergeometric series representation of the generalized kernels

For $x = y = 0$, from Eq. (27),

$$K_{n-1}^{(r,s)}(0,0) = \sum_{j=0}^{n-1} \frac{[L_j^{(\alpha)}(0)]^{(r)}[L_j^{(\alpha)}(0)]^{(s)}}{\|L_j^{(\alpha)}\|_\alpha^2}.$$

Using (28), (41), and (A.1), the previous equation reads

$$K_{n-1}^{(r,s)}(0,0) = \frac{(-1)^{r+s}}{\Gamma(r+\alpha+1)\Gamma(s+\alpha+1)} \sum_{j=0}^{n-1} \frac{j!\,\Gamma(j+\alpha+1)}{(j-r)!\,(j-s)!}.$$

We can assume without loss of generality that $r \geqslant s$. Then, writing $j - r = k$,

$$K_{n-1}^{(r,s)}(0,0) = \frac{(-1)^{r+s}}{\Gamma(r+\alpha+1)\Gamma(s+\alpha+1)} \sum_{k=0}^{n-r-1} \frac{(r+k)!\,\Gamma(r+k+\alpha+1)}{k!\,(r-s+k)!} = \frac{(-1)^{r+s} r!}{(r-s)!\,\Gamma(s+\alpha+1)}\ {}_2F_1\left(\begin{matrix} r+1, r+\alpha+1 \\ r-s+1 \end{matrix}\,\middle|\,1\right)_{n-r-1},$$

where in the second expression we have used Slater's notation for the partial sum of a hypergeometric series [20, p. 83]. Taking advantage of the partial summation theorem [20, p. 82]

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\,1\right)_{N-1} = \frac{\Gamma(a+N)\Gamma(b+N)}{(N-1)!\,\Gamma(c+N)\Gamma(1+a+b-c)} \times {}_3F_2\left(\begin{matrix} c-a, c-b, c+N-1 \\ c, c+N \end{matrix}\,\middle|\,1\right)$$

(note that for $c=b$, $N=m+1$ this formula reduces to (31)), we obtain the more convenient expression

$$K_{n-1}^{(r,s)}(0,0)=\frac{(-1)^{r+s}r!\,n!\,\Gamma(n+\alpha+1)}{(r-s)!\,(n-r-1)!\,(n-s)!\,\Gamma(s+\alpha+1)\Gamma(r+s+\alpha+2)}\times{}_3F_2\left(\left.\begin{matrix}-s,-s-\alpha,n-s\\ r-s+1,n-s+1\end{matrix}\,\right|\,1\right).$$

It is interesting to note that an equivalent expression of $K_{n-1}^{(r,s)}(0,0)$ as a ${}_3F_2$ hypergeometric series has been recently obtained by Bavinck [4] using a different method. In particular, for $s=1$, we have

$$K_{n-1}^{(r,1)}(0,0)=\frac{(-1)^{r+1}\Gamma(n+\alpha+1)[nr+(n-1)(\alpha+1)]}{(n-r-1)!\,\Gamma(\alpha+2)\Gamma(r+\alpha+3)},$$

and setting $r=1$ in this equation, (34) is obtained again.

A.3. Alternative derivation of the connection formula (40)

We consider the Fourier expansion

$$L_n^{(\alpha)}(x;\lambda)=\sum_{j=0}^{n}\rho_{n,j}L_j^{(\alpha+2)}(x)\tag{A.2}$$

with $\rho_{n,n}=1$ due to the monic character of the polynomials. Taking into account that

$$\rho_{n,j}=\frac{\int_0^\infty L_n^{(\alpha)}(x;\lambda)L_j^{(\alpha+2)}(x)x^{\alpha+2}e^{-x}\,dx}{\|L_j^{(\alpha+2)}\|_{\alpha+2}^2}=\frac{\langle L_n^{(\alpha)}(x;\lambda),x^2L_j^{(\alpha+2)}(x)\rangle_{\mathrm{S}}}{\|L_j^{(\alpha+2)}\|_{\alpha+2}^2},$$

we readily notice that $\rho_{n,j}=0$ for $0\leqslant j\leqslant n-3$, so that the expansion (A.2) reduces to

$$L_n^{(\alpha)}(x;\lambda)=L_n^{(\alpha+2)}(x)+\rho_{n,n-1}L_{n-1}^{(\alpha+2)}(x)+\rho_{n,n-2}L_{n-2}^{(\alpha+2)}(x).$$

Taking into account (26), $\rho_{n,n-2}$ can be computed as follows:

$$\begin{aligned}\rho_{n,n-2}&=\frac{\int_0^\infty L_n^{(\alpha)}(x;\lambda)x^2L_{n-2}^{(\alpha+2)}(x)x^{\alpha}e^{-x}\,dx}{\|L_{n-2}^{(\alpha+2)}\|_{\alpha+2}^2}\\&=\frac{1}{\|L_{n-2}^{(\alpha+2)}\|_{\alpha+2}^2}\left(\|L_n^{(\alpha)}\|_\alpha^2+\frac{\lambda([L_n^{(\alpha)}(0)]')^2}{1+\lambda K_{n-1}^{(1,1)}(0,0)}\right)\\&=\frac{1}{\|L_{n-2}^{(\alpha+2)}\|_{\alpha+2}^2}\left(\|L_n^{(\alpha)}\|_\alpha^2\frac{1+\lambda K_n^{(1,1)}(0,0)}{1+\lambda K_{n-1}^{(1,1)}(0,0)}\right)\\&=n(n-1)\frac{1+\lambda K_n^{(1,1)}(0,0)}{1+\lambda K_{n-1}^{(1,1)}(0,0)}=n(n-1)\lambda_n.\end{aligned}$$

On the other hand,

$$\rho_{n,n-1} = \frac{\int_0^\infty L_n^{(\alpha)}(x;\lambda)x^2 L_{n-1}^{(\alpha+2)}(x)x^\alpha e^{-x}\,dx}{\|L_{n-1}^{(\alpha+2)}\|_{\alpha+2}^2}. \tag{A.3}$$

Now let us make use of the structure relation for Laguerre polynomials,

$$x\big[L_n^{(\alpha)}(x)\big]' = nL_n^{(\alpha)}(x) + n(n+\alpha)L_{n-1}^{(\alpha)}(x). \tag{A.4}$$

Differentiating this equality, we readily fi d that

$$x\big[L_n^{(\alpha)}(x)\big]'' = (n-1)\big[L_n^{(\alpha)}(x)\big]' + n(n+\alpha)\big[L_{n-1}^{(\alpha)}(x)\big]'.$$

If we now multiply both sides of this equality by x and use again the structure relation (A.4), we obtain

$$\begin{aligned}x^2\big[L_n^{(\alpha)}(x)\big]'' &= (n-1)nL_n^{(\alpha)}(x) + 2(n-1)n(n+\alpha)L_{n-1}^{(\alpha)}(x)\\ &\quad + n(n+\alpha)(n-1)(n+\alpha-1)L_{n-2}^{(\alpha)}(x).\end{aligned}$$

Recalling that $[L_n^{(\alpha)}(x)]'' = n(n-1)L_{n-2}^{(\alpha+2)}(x)$, the previous equation can be written as

$$x^2 L_{n-2}^{(\alpha+2)}(x) = L_n^{(\alpha)}(x) + 2(n+\alpha)L_{n-1}^{(\alpha)}(x) + (n+\alpha)(n+\alpha-1)L_{n-2}^{(\alpha)}(x),$$

or, equivalently,

$$x^2 L_{n-1}^{(\alpha+2)}(x) = L_{n+1}^{(\alpha)}(x) + 2(n+\alpha+1)L_n^{(\alpha)}(x) + (n+\alpha)(n+\alpha+1)L_{n-1}^{(\alpha)}(x).$$

Thus the numerator in (A.3) becomes

$$\begin{aligned}&2(n+\alpha+1)\big\|L_n^{(\alpha)}\big\|_\alpha^2 - \frac{\lambda[L_n^{(\alpha)}(0)]'}{1+\lambda K_{n-1}^{(1,1)}(0,0)}(n+\alpha)(n+\alpha+1)\big[L_{n-1}^{(\alpha)}(0)\big]'\\ &\quad = (n+\alpha+1)\left(2\big\|L_n^{(\alpha)}\big\|_\alpha^2 + \frac{\lambda([L_n^{(\alpha)}(0)]')^2}{1+\lambda K_{n-1}^{(1,1)}(0,0)}\frac{(n-1)}{n}\right),\end{aligned}$$

according to (26). After some algebra, we fin that

$$\rho_{n,n-1} = (n+1) + (n-1)\frac{1+\lambda K_n^{(1,1)}(0,0)}{1+\lambda K_{n-1}^{(1,1)}(0,0)} = (n+1) + (n-1)\lambda_n,$$

which completes the proof of (40).

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